Effective Impedance of a Dynamic Network; Spectral Properties, and Application

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Outline

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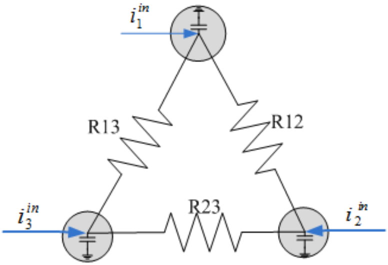
Electrical Networks and Static Graphs



- The static Laplacian matrix $L = [l_{ij}]$ is defined by:

$$l_{ij} = \begin{cases} \sum_{j \in \mathcal{N}_i} \frac{1}{R_{ij}} & i = j \\ -\frac{1}{R_{ij}} & i \neq j \text{ and } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Properties of the Static Laplacian Matrix

- For a graph G and its Laplacian matrix $L \in \mathbb{R}^{n\times n}$ with eigenvalues $(\lambda_1(L) \leq \lambda_2(L) \leq ... \leq \lambda_n(L))$:
  1. L is always **positive-semidefinite** ($\forall i,\ \lambda_i \geq 0,\ \lambda_1 = 0$)
  2. The row sums of L are all zero
  3. L is diagonally dominant
  4. $\lambda_1(L) = 0$ with eigenvector $\mathbf{1}$
- If the graph G is connected:
  1. $\lambda = 0$ is a distinct eigenvalue of L
  2. If $r^T L = 0$ (i.e., r is a left eigenvector of L), scaled so that $r^T\mathbf{1} = 1$ then

$$\lim_{t\to\infty} e^{Lt} = \mathbf{1}r^T$$

  3. For a vector $x = [x_1, x_2, \ldots, x_N]^T$, the solution of $\dot{x} = \Lambda x$ satisfies $x_i \to x^*$ for some constant x^* (i.e., consensus!)

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Consensus Problems over Graphs

- More generally, we can define several classes of problems

Case	Nodes	Arcs (Edges)	Problem Type
1	no processing	static weighted arcs	normal graph
2	integrating nodes	static weighted arcs	consensus problem
3	integrating nodes	dynamic arcs	dynamic consensus
4	dynamic nodes	dynamic arcs	most general

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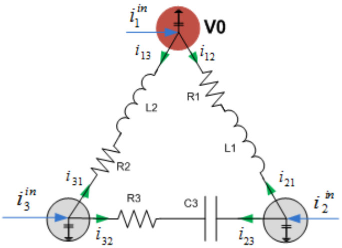
- Electrical network as a dynamic network



- Using the Kirchhoff's law. The dynamic of each node is obtained as

$$C\frac{dv_1}{dt} = \dot{v_1} = i_1^{in} - i_1^{out};$$
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where,

$$i_1^{out} = i_{12} + i_{13};$$
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$$i_{uv} = \frac{1}{Z_{uv}(s)}(v_u - v_v).\forall u, v \in 1, 2, 3;$$

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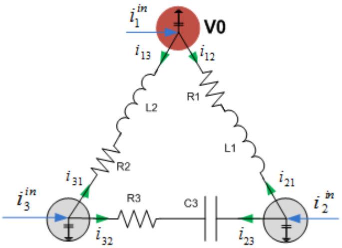

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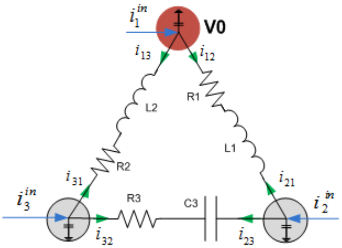

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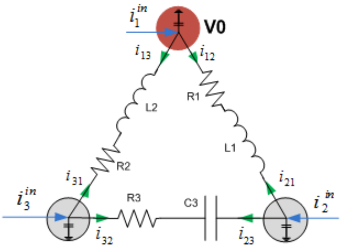

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Electrical Networks and Dynamic Graph

- The output current for a nod $u \in 1, 2, 3$ can be written as

$$i_u^{out} = \sum_{v \in \mathcal{N}_u} \frac{1}{Z_{uv}} (v_u - v_v) = \sum_{v \in \mathcal{N}_u} Y_{uv} (v_u - v_v),$$

  where, $Y_{uv} = \frac{1}{Z_{uv}}$ is the admittance between the nodes u, v.

- From the last equations, we can write the relationships between the node potentials and the output currents as

$$\begin{bmatrix} i_1^{out} \\ i_2^{out} \\ i_3^{out} \end{bmatrix} = \begin{bmatrix} \frac{1}{Z_{12}(s)} + \frac{1}{Z_{13}(s)} & -\frac{1}{Z_{12}(s)} & -\frac{1}{Z_{13}(s)} \\ -\frac{1}{Z_{21}(s)} & \frac{1}{Z_{21}(s)} + \frac{1}{Z_{23}(s)} & -\frac{1}{Z_{23}(s)} \\ -\frac{1}{Z_{31}(s)} & -\frac{1}{Z_{32}(s)} & \frac{1}{Z_{31}(s)} + \frac{1}{Z_{32}(s)} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = L(j\omega) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

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$$L(j\omega) = \begin{bmatrix} \frac{1}{Z_{12}(j\omega)} + \frac{1}{Z_{13}(j\omega)} & -\frac{1}{Z_{12}(j\omega)} & -\frac{1}{Z_{13}(j\omega)} \\ -\frac{1}{Z_{21}(j\omega)} & \frac{1}{Z_{21}(j\omega)} + \frac{1}{Z_{23}(j\omega)} & -\frac{1}{Z_{23}(j\omega)} \\ -\frac{1}{Z_{31}(j\omega)} & -\frac{1}{Z_{32}(j\omega)} & \frac{1}{Z_{31}(j\omega)} + \frac{1}{Z_{32}(j\omega)} \end{bmatrix}$$

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- For the autonomous case ($i_u^{in} = 0$), the dynamic of each node

$$\dot{v_i} = -\sum_{j \in \mathcal{N}_i} \frac{1}{Z_{ij}} (v_i - v_j), \forall i = 1, 2, 3$$

- The overall system can be represented by

$$\dot{v(t)} = -L(\frac{d}{dt}) v(t)$$

- Taking the Laplace transform on both sides we get

$$sV(s) - v(0) = -L(s)V(s);$$

$$V(s) = (sI_n + L(s))^{-1} v(0);$$

  - Compared with the **static case**:

$$V(s) = (sI_n + L)^{-1} v(0);$$

- If $L(0) = L$, then the consensus value $\alpha = \sum_i v_i(0)/n$

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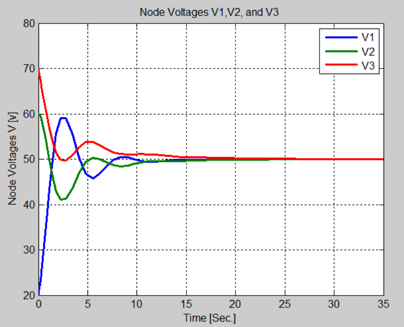
Simulation Result

- Dynamic Laplacian $L(s)$ and Static Laplacian $L(0)$

$$L(s) = \begin{bmatrix} \frac{1}{2s+1} + \frac{1}{2s+3} & -\frac{1}{2s+1} & -\frac{1}{2s+3} \\ -\frac{1}{2s+1} & \frac{s}{s+1} + \frac{1}{2s+1} & -\frac{s}{s+1} \\ -\frac{1}{2s+3} & -\frac{s}{s+1} & \frac{s}{s+1} + \frac{1}{2s+3} \end{bmatrix}; L(0) \quad = \begin{bmatrix} 1 + \frac{1}{3} & -1 & -\frac{1}{3} \\ -1 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$



The conditions of the arcs in a dynamic graphs

Definition

$Z_{ij}(s)$ is positive real (PR) if $Re[Z_{ij}(s)] \geq 0$, $\forall Re[s] > 0$.

The arcs $Z_{ij}(s)$ must satisfy the following conditions

1. $Z_{ij}(s) = Z_{ji}(s), i, j \in V(G)$,
2. $Z_{ij}(s) \neq 0$ if and only if i and j are adjacent in G,
3. $Z_{ij}(s)$ is positive real (PR), $i, j \in V(G)$.

Positive Definiteness of a Complex Matrix [Johnson 1970]

Def.

An $n \times n$ complex matrix A is called positive definite PD (respectively, positive semidefinite PSD) if $Re[x^H Ax] > 0$ (respectively, $Re[x^H Ax] \geq 0$) for all complex vector $x \in \mathbb{C}^n$, where x^H denotes the conjugate transpose of the vector x.

Lemma 1.

A necessary and sufficient condition for a complex matrix A to be PD (respectively, PSD) is that the Hermitian part $H(A) = \frac{1}{2}(A + A^H)$, be PD (respectively, PSD).

Fact

An important sufficient condition for a matrix to be positive stable (all eigenvalues have positive real parts) is the following fact: Let $A \in \mathbb{C}^{n \times n}$. If $A + A^H$ is PD, then A is positive stable.

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Lemma 2.

A necessary and sufficient condition for the dynamic Laplacian $L(j\omega)$ to be a PSD matrix is that the real part of $L(j\omega)$ be a PSD matrix.

Proof.

- For $L(j\omega) \in \mathbb{C}^{n\times n}$, $L(j\omega) = H(L(j\omega)) + S(L(j\omega))$, where $H(L(j\omega)) = \frac{1}{2}(L(j\omega) + L(j\omega)^H)$ denotes the Hermitian part of $L(j\omega)$ and $S(L(j\omega)) = \frac{1}{2}(L(j\omega) - L(j\omega)^H)$ denotes the skew-Hermitian part of $L(j\omega)$.
- By definition, $L(j\omega)$ is symmetric matrix, then

$$L(j\omega)^H = \overline{L(j\omega)}^T = \overline{L(j\omega)};$$

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- Based on Lemma 1., we can conclude that if the real part of the dynamic Laplacian $Re[L(j\omega)]$ is PSD then $L(j\omega)$ is PSD.

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Lemma

The real part of the dynamic Laplacian matrix $Re[L(j\omega)]$ is PSD matrix and all principal sub matrices of $Re[L(j\omega)]$ are PD.

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- From the definition of the dynamic Laplacian we can write $Re[L(j\omega)] = [l_{ij}]$ as

$$l_{ij} = \begin{cases} \sum_{j \in \mathcal{N}_i} Re[\frac{1}{Z_{ij}(j\omega)}] & i = j \\ -Re[\frac{1}{Z_{ij}(j\omega)}] & i \neq j \text{ and } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

- Since $Re[\frac{1}{Z_{ij}(j\omega)}]$ is PR, thus $Re[L(j\omega)]$ is real symmetric (static Laplacian) matrix. So, it is PSD and all sub principal sub matrices are PD.

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Example

- The dynamic Laplacian for the electrical network with 3 nodes and the same impedance for each edge $Z_{ij} = R + j\omega L$, $R = 1\Omega$, and $L = 1H$ is given by:

$$L(j\omega) = \begin{bmatrix} \frac{2}{1+j\omega} & -\frac{1}{1+j\omega} & -\frac{1}{1+j\omega} \\ -\frac{1}{1+j\omega} & \frac{2}{1+j\omega} & -\frac{1}{1+j\omega} \\ -\frac{1}{1+j\omega} & -\frac{1}{1+j\omega} & \frac{2}{1+j\omega} \end{bmatrix};$$

$$Re[L(j\omega) = \frac{1}{1+\omega^2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}; \rightarrow (Real - Symmetric - PSD - matrix)$$

Properties of the Dynamic Laplacian

Lemma 3

Let G be a dynamic graph with all arcs positive real (PR). Then:

1. The dynamic Laplacian $L(G)$ is complex symmetric positive semidefinite (CSPSD),
2. The real part of eigenvalues of $L(G)$ are non-negative $(Re[\lambda_i(L(j\omega))] \geq 0)$ $\forall i \in 1, 2, ..., n)$,

$$0 = \lambda_1(L(j\omega)) < Re[\lambda_2(L(j\omega))] \leq Re[\lambda_3(L(j\omega))] ... \leq Re[\lambda_n(L(j\omega))]$$

3. $\lambda_1(L(j\omega)) = 0$ with eigenvector $\mathbf{1}$.

Proof

1) The dynamic Laplacian is CSPSD matrix

- Since $Z_{ij}(j\omega) = Z_{ji}(j\omega)$, Thus $L(j\omega)$ is CS and $H(L(j\omega)) = Re[L(j\omega]$
- In consideration of the positivity realness of the arcs, then $Re[L(j\omega]$ matrix is the static Laplacian matrix, $Re[L(j\omega]$ is PSD.
- Based on Lemma 2, the dynamic Laplacian matrix is CSPSD matrix.

2) The real part of the eigenvalues of $L(j\omega)$ are non-negative

- By definition, if $L(j\omega)$ is PSD then $Re[x^H Ax] \geq 0$ for all complex vector $x \in \mathbb{C}^n$
- In particular, is true for $x = v_i$, where v_i is the i-th eigenvector of $L(j\omega)$

$$Re[v_i^H L(j\omega)v_i] \geq 0$$

- From the definition of the eigenvalues and eigenvectors ($L(j\omega)v_i = \lambda_i v_i$), we can write the last inequality as

$$Re[v_i^H \lambda_i v_i] \geq 0$$

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Proof Cont.

$$Re[v_i^H \lambda_i v_i] \geq 0 \Rightarrow Re[\lambda_i v_i^H v_i] \geq 0 \Rightarrow Re[\lambda_i \|v_i\|_2] \geq 0$$

- Since $\|v_i\|_2 > 0$, then $Re[\lambda_i] \geq 0$.

3) $\lambda_1(L(j\omega)) = 0$ with eigenvector $\mathbf{1}$.

- From the definition of $L(j\omega)$, we can observe that the rows of $L(j\omega)$ sum to zero, which implies that $L(j\omega)x = 0$ if all the entires of x are the same, so x is the eigenvector of eigenvalue 0.

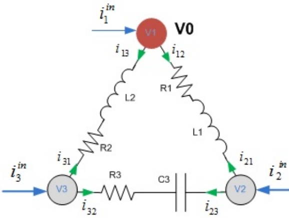
Outline

Effective Impedance

- The effective impedance of a node $u \in 2, 3$ to a node $V_0 = 1$, denoted by $Z_u^{eff}(V_0)(j\omega)$ can be defined as

$$Z_2^{eff}(1)(j\omega) = \frac{v_2 - v_1}{i_2^{in}}|_{v_1=0, i_3^{out}=0} = \frac{v_2}{i_2^{out}}|_{v_1=0, i_3^{out}=0};$$

$$Z_3^{eff}(1)(j\omega) = \frac{v_3 - v_1}{i_3^{in}}|_{v_1=0, i_2^{out}=0} = \frac{v_3}{i_3^{out}}|_{v_1=0, i_2^{out}=0}.$$



- The dynamic Laplacian describes the relationship between currents and voltages as

$$\begin{bmatrix} i_1^{out} \\ i_2^{out} \\ i_3^{out} \end{bmatrix} = \begin{bmatrix} \frac{1}{Z_{12}(s)} + \frac{1}{Z_{13}(s)} & -\frac{1}{Z_{12}(s)} & -\frac{1}{Z_{13}(s)} \\ -\frac{1}{Z_{21}(s)} & \frac{1}{Z_{21}(s)} + \frac{1}{Z_{23}(s)} & -\frac{1}{Z_{23}(s)} \\ -\frac{1}{Z_{31}(s)} & -\frac{1}{Z_{32}(s)} & \frac{1}{Z_{31}(s)} + \frac{1}{Z_{32}(s)} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = L(j\omega) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

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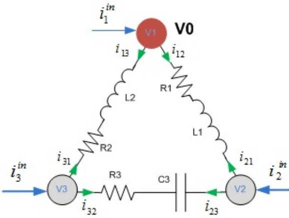

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- Given $v_1 = 0 \Rightarrow i_1^{in} = i_1^{out} = 0$. Using ray transfer matrix we can obtain:

$$\begin{bmatrix} i_2^{out} \\ i_3^{out} \end{bmatrix} = \begin{bmatrix} \frac{1}{Z_{21}(s)} + \frac{1}{Z_{23}(s)} & -\frac{1}{Z_{23}(s)} \\ -\frac{1}{Z_{32}(s)} & \frac{1}{Z_{31}(s)} + \frac{1}{Z_{32}(s)} \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} i_2^{out} \\ i_3^{out} \end{bmatrix} = L_0(j\omega) \begin{bmatrix} v_2 \\ v_3 \end{bmatrix};$$

where, $L_0(j\omega)$ is the Ground Laplacian.

- Ray Transfer Matrix

$$\begin{pmatrix} x_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix},$$

where

$$A = \left.\frac{x_2}{x_1}\right|_{\theta_1=0} \qquad B = \left.\frac{x_2}{\theta_1}\right|_{x_1=0},$$

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Effective Impedance

$$Z_2^{eff}(1)(j\omega) = \frac{v_2}{i_2^{out}}|_{v_1=0, i_3^{out}=0} = L_0(j\omega)^{-1}(1,1)$$

$$Z_3^{eff}(1)(j\omega) = \frac{v_3}{i_3^{out}}|_{v_1=0, i_2^{out}=0} = L_0(j\omega)^{-1}(2,2)$$

$$\sum_{u \in \mathbf{V}} Z_u^{eff}(V_0)(j\omega) = trac(L0(j\omega)^{-1}).$$

- **Dynamic Ground Laplacian** $L_0(j\omega)$ is obtained from the dynamic Laplacian matrix $L(j\omega) \in \mathbb{R}^{n \times n}(s)$ by removing all rows and columns corresponding to the nodes in $\mathbf{V_0}$.
- **Effective Impedance**
  Given a dynamic graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{V}$ is a set of n nodes; $\mathbf{E} \subset \mathbf{V} \times \mathbf{V}$ a set of m edges, and given a subset $\mathbf{V_0} \subset \mathbf{V}$ consisting of $n_0 < n$ nodes, the *effective impedance* of a node $u \in V$ to $\mathbf{V_0}$, denoted by $Z_u^{eff}(\mathbf{V_0})(j\omega)$, is the element in the main diagonal of $L_0^{-1}(j\omega)$ associated with the node $u \in \mathbf{V}$.

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Properties of the Ground Laplacian $L_0(j\omega)$

Lemma 4

The Ground Laplacian $L_0(j\omega)$ is CSPD matrix and always invertible for all ω.

Proof

- Since $Z_{ij}(j\omega) = Z_{ji}(j\omega)$, Thus $L_0(j\omega)$ is CS matrix.
- $L_0(j\omega)$ is PD because the Hermitian part of $L_0(j\omega)$

$$H(L_0(j\omega)) = \frac{1}{2}(L_0(j\omega) + L_0(j\omega)^H) = \frac{1}{2}(L_0(j\omega) + \overline{L_0(j\omega)}) = Re[L_0(j\omega)]$$

is PD.

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Outline

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The lower bounds for the $\lambda_i(H(L_0(j\omega)))$

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The lower bonds of the eigenvalues of the Hermitian part of the dynamic Ground Laplacian matrix can be obtained from the Ground Laplacian as

$$\lambda_i(H(L_0(j\omega))) \geq \frac{1}{trace((Re[L_0(j\omega)])^{-1})}, \forall\omega, i \in 1, 2, ..., n.$$

proof

- We know that the sum of the eigenvalues of any matrix is equal to its trace

$$\sum_{i=1}^{n} \lambda_i(H(L_0(j\omega))) = trace(H(L_0(j\omega)))$$

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Proof Cont.

- Since $H(L_0(j\omega))$ is real PD matrix, then $\lambda_i(H(L_0(j\omega))) > 0$,

$$\lambda_i(H(L_0(j\omega))) \leq trace(H(L_0(j\omega)) = trace(Re[L_0(j\omega)])$$

- We know that $H(L_0(j\omega))$ is real PD matrix and invertible, then $[H(L_0(j\omega))]^{-1}$ is also PD matrix,

$$\lambda_i([H(L_0(j\omega))]^{-1}) \leq trace([H(L_0(j\omega)]^{-1}) = trace((Re[L_0(j\omega)])^{-1})$$

- Since the eigenvalues of $H(L_0(j\omega))$ and $[H(L_0(j\omega))]^{-1}$ are reciprocals of each other: $H(L_0(j\omega)v_i = \lambda_i(H(L_0(j\omega)))v_i \Rightarrow \frac{1}{\lambda_i(H(L_0(j\omega)))}v_i = [H(L_0(j\omega))]^{-1}v_i$, then

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Proof Cont.

- Since $H(L_0(j\omega))$ is real PD matrix, then $\lambda_i(H(L_0(j\omega))) > 0$,

$$\lambda_i(H(L_0(j\omega))) \leq trace(H(L_0(j\omega)) = trace(Re[L_0(j\omega)])$$

- We know that $H(L_0(j\omega))$ is real PD matrix and invertible, then $[H(L_0(j\omega))]^{-1}$ is also PD matrix,

$$\lambda_i([H(L_0(j\omega))]^{-1}) \leq trace([H(L_0(j\omega)]^{-1}) = trace((Re[L_0(j\omega)])^{-1})$$

- Since the eigenvalues of $H(L_0(j\omega))$ and $[H(L_0(j\omega))]^{-1}$ are reciprocals of each other: $H(L_0(j\omega)v_i = \lambda_i(H(L_0(j\omega)))v_i \Rightarrow \frac{1}{\lambda_i(H(L_0(j\omega)))}v_i = [H(L_0(j\omega))]^{-1}v_i$, then

$$\lambda_i([H(L_0(j\omega))]^{-1}) = \frac{1}{\lambda_i(H(L_0(j\omega)))}$$

Proof Cont.

- Substituting this result in the last inequality, we get

$$\lambda_i(H(L_0(j\omega))) \geq \frac{1}{trace((Re[L_0(j\omega)])^{-1})}, \forall\omega, i \in 1, 2, ..., n.$$

The relationship between $\lambda_i(H(L(j\omega)))$ and $\lambda_i(H(L_0(j\omega)))$

Cauchy Interlace Theorem

Let A be a Hermitian matrix of order n, and let B be a principal sub matrix of A of order $n-1$. if $\lambda_{min} = \lambda_n \leq \lambda_{n-1} \leq ... \leq \lambda_2 \leq \lambda_1 = \lambda_{max}$ lists the eigenvalues of A and $\mu_n \leq \mu_{n-1} \leq ... \leq \mu_3 \leq \mu_2$ the eigenvalues of B. then

$$\lambda_n \leq \mu_n \leq \lambda_{n-1} \leq \mu_{n-1} \leq ... \leq \lambda_2 \leq \mu_2 \leq \lambda_1$$

- Applying the Interlacing Theorem for the matrices $H(L(j\omega))$ and $H(L_0(j\omega))$, we can obtain the inequality that relates $\lambda_i(H(L(j\omega)))$ and $\lambda_i(H(L_0(j\omega)))$ as

$$\lambda_{i+1}(H(L(j\omega))) \geq \lambda_i(H(L_0(j\omega)))\forall i = 1, 2, ..., n-1.$$

-**Note:** The eigenvalues are arranged in algebraically increasing order.

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The relationship between $\lambda_i(L(j\omega))$ and $\lambda_i(H(L(j\omega)))$

L. Mirsky Theorem

Let $A \in \mathbb{C}^{n\times n}$ be Given. The one of the natural Hermitian matrices associated with A: $H(A) = \frac{1}{2}(A + A^H)$. A Theorem of L. Mirsky characterizes the relationship between the eigenvalues of A and H(A). Let $\lambda_i(A)$ and $\lambda_i(H(A))$ denote the eigenvalues of A and H(A), respectivaly, ordered so that $Re[\lambda_1(A)] \geq ... \geq Re[\lambda_n(A)]$ and $\lambda_1(H(A)) \geq ... \geq \lambda_n(H(A))$. Then

$$\sum_{i=1}^{k} Re[\lambda_i(A)] \leq \sum_{i=1}^{k} \lambda_i(H(A)),$$

k=1,2,...,n, with equality for k=n.

The relationship between $\lambda_i(L(j\omega))$ and $\lambda_i(H(L(j\omega)))$

Proposition 1

Let the dynamic Laplacian $L(j\omega) \in \mathbb{C}^{n\times n}$ be Given. The Hermitian matrix associated with $L(j\omega)$: $H(L(j\omega)) = \frac{1}{2}(L(j\omega) + \overline{L(j\omega)}) = Re[L(j\omega)]$. Let $\lambda_i(L(j\omega))$ and $\lambda_i(H(L(j\omega)))$ denote the eigenvalues of $L(j\omega)$ and $H(L(j\omega))$, respectively, ordered so that $0 = Re[\lambda_1(L(j\omega))] < Re[\lambda_n(L(j\omega))] ... \leq Re[\lambda_n(L(j\omega))]$ and $0 = \lambda_1(H(L(j\omega))) < \lambda_2(H(L(j\omega))) ... \leq \lambda_n(H(L(j\omega)))$. Then the relationship between the real part of the smallest and largest nonzero eigenvalues of $L(j\omega)$ and $H(L(j\omega))$ can be given by

$$Re[\lambda_2(L(j\omega))] \geq \lambda_2(H(L(j\omega)))$$

$$Re[\lambda_n(L(j\omega))] \leq \lambda_n(H(L(j\omega)))$$

Proof of the Proposition 1

Proof

- Applying the Theorem of L. Mirskey for $k = n - 2$, yields

$$\sum_{i=1}^{n-2} Re[\lambda_i(L(j\omega))] \leq \sum_{i=1}^{n-2} \lambda_i(H(L(j\omega))), \quad (1)$$

- Note: the eigenvalues $Re[\lambda_i(L(j\omega))]$ and $\lambda_i(H(L(j\omega)))$ in this Theorem are ordered in the decreasing order.

- For $k = n$, we could obtain the following quality

$$\sum_{i=1}^{n} Re[\lambda_i(L(j\omega))] = \sum_{i=1}^{n} \lambda_i(H(L(j\omega))),$$

- For a connected graph, $Re[\lambda_n(L(j\omega))] = \lambda_n(H(L(j\omega))) = 0$, thus we can write the last inequality as

$$\sum_{i=1}^{n-2} Re[\lambda_i(L(j\omega))] + Re[\lambda_{n-1}(L(j\omega))] + 0 = \sum_{i=1}^{n-2} \lambda_i(H(L(j\omega))) + \lambda_{n-1}(H(L(j\omega))) + 0 \quad (2)$$

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- For a connected graph, $Re[\lambda_n(L(j\omega))] = \lambda_n(H(L(j\omega))) = 0$, thus we can write the last inequality as

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Proof Cont.

- From (1) and (2), we can conclude

$$Re[\lambda_{n-1}(L(j\omega))] \geq \lambda_{n-1}(H(L(j\omega)))$$

- If we consider the increasing order of $Re[\lambda_i(L(j\omega))]$ and $\lambda_i(H(L(j\omega)))$ then

$$Re[\lambda_2(L(j\omega))] \geq \lambda_2(H(L(j\omega)))$$

- Now, for $k = 1$

$$Re[\lambda_1(L(j\omega))] \leq \lambda_1(H(L(j\omega))),$$

Since we are considering the increasing order of the eigenvalues, then

$$Re[\lambda_n(L(j\omega))] \leq \lambda_n(H(L(j\omega))).$$

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The Lower Bounds for the Smallest Non-Zero Eigenvalues of the Dynamic Laplacian

Lemma 6

Let the dynamic Laplacian matrix $L(j\omega) \in \mathbb{C}^{n\times n}$ be given. Let $\lambda_i(L(j\omega))$ and $\lambda_i(L_0(j\omega))$ denote the eigenvalues of $L(j\omega)$ and $L_0(j\omega)$, respectively, ordered so that $Re[\lambda_1(L(j\omega))] \leq Re[\lambda_2(L(j\omega))] \leq ... \leq Re[\lambda_n(L(j\omega))]$ and $Re\lambda_1(L_0(j\omega)) \leq Re\lambda_2(L_0(j\omega)) \leq ... \leq Re\lambda_{n_0}(L_0(j\omega))$, $n_0 < n$, then The lower bounds for the real part of the smallest non-zero eigenvalues of the dynamic Laplacian is given by

$$Re[\lambda_2^{min}(L(j\omega))] \geq \frac{1}{\|trace((Re[L_0(j\omega)])^{-1})\|_\infty}.$$

Proof

- Based on Lemma 5, the lower bonds for the eigenvalues of $H(L_0(j\omega))$ can be obtained from the Ground Laplacian matrix as

$$\lambda_i(H(L_0(j\omega))) \geq \frac{1}{trace((Re[L_0(j\omega)])^{-1})}, \forall \omega, i \in 1, 2, ..., n.$$

- Applying the Interlacing Theorem, we can conclude that the eigenvalues of $H(L(j\omega)) \in \mathbb{R}^{n\times n}$ and $H(L_0(j\omega)) \in \mathbb{R}^{n-n_0\times n-n_0}$ are interlaced for $i = 1, 2, ..., n - n_0$

$$\lambda_i(H(L(j\omega))) \geq \lambda_i(H(L_0(j\omega))) \geq \lambda_{i+n_0}(H(L(j\omega))),$$

$$\lambda_i(H(L(j\omega))) \geq \frac{1}{trace((Re[L_0(j\omega)])^{-1})},$$

$$\Rightarrow \lambda_2(H(L(j\omega))) \geq \frac{1}{trace((Re[L_0(j\omega)])^{-1})}.$$

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$$\Rightarrow \lambda_2(H(L(j\omega))) \geq \frac{1}{trace((Re[L_0(j\omega)])^{-1})}.$$

- Based on Proposition 1, the smallest non zero eigenvalues of $L(j\omega)$ can be written as

$$Re[\lambda_2(L(j\omega))] \geq \lambda_2(H(L(j\omega))),$$

hence

$$Re[\lambda_2(L(j\omega))] \geq \frac{1}{trace((Re[L_0(j\omega)])^{-1})}.$$

- Taking the minimum values of both sides in the above inequality over all ω,

$$\min_{\omega} Re[\lambda_2(L(j\omega))] \geq \min_{\omega} \frac{1}{trace((Re[L_0(j\omega)])^{-1})},$$

$$Re[\lambda_2^{min}(L(j\omega))] \geq \frac{1}{\max_{\omega} trace((Re[L_0(j\omega)])^{-1})},$$

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The Upper Bounds for the Largest Eigenvalues of the Dynamic Laplacian

Lemma

Let the dynamic Laplacian matrix $L(j\omega) \in \mathbb{C}^{n\times n}$ be given. Let $\lambda_i(L(j\omega))$ denotes the eigenvalues of $L(j\omega)$, ordered so that $Re[\lambda_1(L(j\omega))] \leq Re[\lambda_2(L(j\omega))] \leq ... \leq Re[\lambda_n(L(j\omega))]$, then the upper bounds for the real part of the largest eigenvalues of the dynamic Laplacian is given by

$$\Rightarrow Re[\lambda_n^{max}(L(j\omega))] \leq 2max_i \left\| Re[D(j\omega)(i,i)] \right\|_\infty .$$

Proof.

- In the static graph (the edges are static), the matrix $(2D - L)$ is PSD matrix. **[Barooah]**

$$2D - L \geq 0$$

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- In the dynamic graph (the edges are dynamic), it can be easily seen that $H(2D(j\omega) - L(j\omega)) = Re[2D(j\omega) - L(j\omega)]$ is also PSD matrix

$$H(2D(j\omega) - L(j\omega)) \geq 0,$$

$$\Rightarrow H(L(j\omega)) \leq 2H(D(j\omega)).$$

- From the above inequality, we can conclude

$$\lambda_i(H(L(j\omega))) \leq 2\lambda_i(H(D(j\omega))) = 2Re[D(j\omega)(i,i)],$$

$$\Rightarrow \lambda_n(H(L(j\omega))) \leq 2Re[D(j\omega)(i,i)].$$

- Based on Proposition 1, we have

$$Re[\lambda_n(L(j\omega))] \leq \lambda_n(H(L(j\omega))),$$

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$$\max_{\omega} Re[\lambda_n(L(j\omega))] \leq 2 \max_{\omega}(Re[D(j\omega)(i,i)]),$$

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Thanks for your attention!
Any Questions?