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## RESEARCH ARTICLE

# Robust $H_\infty$ Controller Design for Dynamic Consensus Networks

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This paper studies an $H_\infty$ suboptimal control problems of consensus networks whereby the weights of network edges are no longer static gains, but instead are dynamic systems, leading to the notion of dynamic consensus networks. We apply model, orthogonal and diagonal transformations to a dynamic consensus network in order to reduce the overall system into $N-1$ independent subsystems. We then establish a generalized methodology for designing a controller for a dynamic consensus network in the presence of external disturbances, focusing especially on using decentralized controllers that achieve consensus in the absence of disturbances and attenuation of disturbances to a prescribed $H_\infty$ performance level. A design example is given to illustrate our results.

**Keywords:** Consensus, dynamic networks, $H_\infty$ suboptimal control.

## 1 Introduction

The idea of consensus in networking has received great attention due to its wide array of applications in fields such as robotics, transportation, sensor networking, communication networking, biology, and physics. The focus of this paper is to study a generalization of consensus problems whereby the weights of network edges are no longer static gains, but instead are dynamic systems, leading to the notion of *dynamic consensus networks.*

### 1.1 *Static Consensus Protocol*

The network topology of this type of problem is static, meaning that there are no dynamics in the interconnections between the nodes ($\lambda_{ij} = constant \geq 0$) and the nodes are assumed to be integrators (Olfati-Saber and Murray 2004). Thus, static consensus problems can be written in

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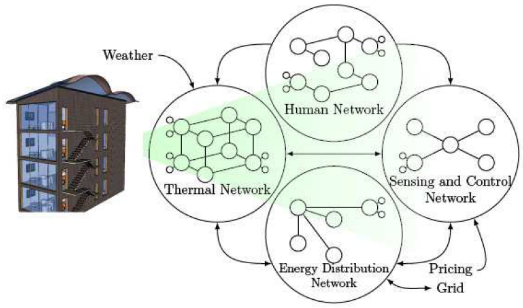


Figure 1: A building as a collection of interacting networks.

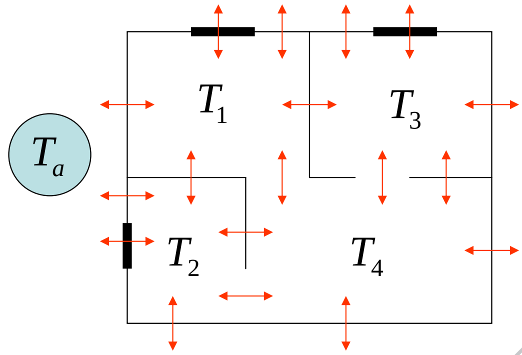


Figure 2: A hypothetical four-room example.

the time domain for each node $i = 1, 2, ..., n$ as

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} \lambda_{ij}(x_j(t) - x_i(t)). \tag{1}$$

The continuous time linear consensus protocol (1) can be written in matrix form as:

$$\dot{x}(t) = -Lx(t), \tag{2}$$

where $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ and $L$, the graph's Laplacian matrix $L = [l_{ij}]$, is defined by

$$l_{ij} = \begin{cases} \sum_{j \in \mathcal{N}_i} \lambda_{ij} & i = j \\ -\lambda_{ij} & i \neq j \text{ and } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

For the multi-agent consensus problem, suppose that $N$ agents evolve their individual beliefs $x_i \in \mathbb{R}^1$ about a so-called global consensus variable $x$ using communications with their nearest neighbors according to the consensus protocol (1). A key result is that the solution of $\dot{x}(t) = -Lx(t)$ gives $x_i \to x^*$ if the static graph is connected (Olfati-Saber et al. 2007). This specific fact has been the basis of much of the literature related to consensus problems.

### 1.2 *Consensus over Networks with Dynamic Edges*

In our recent work we have considered *dynamic consensus networks*, motivated by modeling a thermal process in a building as a directed dynamic graph (Lashhab 2012, Moore et al. 2011). In this work, we present a detailed study of modeling thermal processes in buildings as directed, dynamic graphs, beginning with a simple two-room model and transitioning to a model with multiple interconnected rooms.

Before proceeding, we note that original interest in modeling thermal processes in a building comes from viewing a building as a group of overlapping, interacting networks as shown in Fig. Figure 1[1].

This figure depicts the dominant phenomena that contribute to the energy use of a building as networks composed of aggregate nodes that each represent a distinct subsystem. In the thermal and human networks, the nodes may represent rooms, while in the control network, a node is a sensor, actuator, or computational unit. The links between nodes indicate variable information

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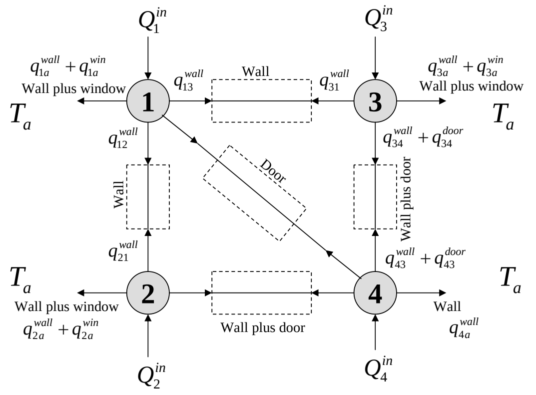


Figure 3: Heat flow network corresponding to the four-room example.

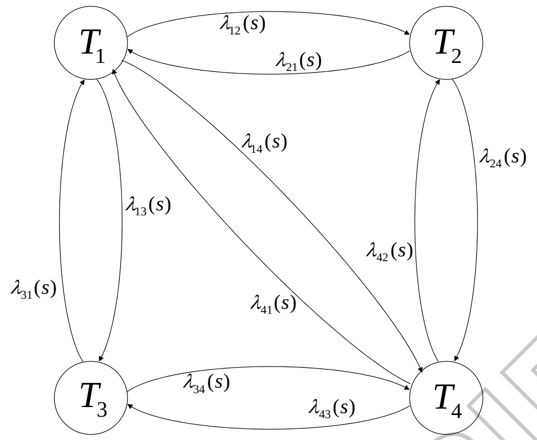

Figure 4: A hypothetical four-room example as a dynamic consensus network.

sharing, such as the flow of heat between rooms through walls and doors in the thermal network. The smaller circles in Fig. Figure 1 indicate links between networks. While typical, graph-based networks assume links that are in some way constant, some networks, such as a building's thermal network, may have dynamic links between nodes, as we will see in the remainder of this section.

The hypothetical four-room building is shown in Fig. Figure 2 in which each room has several neighbors with which it is interconnected. One such neighbor is always the external environment whose variable is denoted $T_a$ with $'a'$ referring to the ambient. Pathways include walls, doors, and windows. The interconnection between the two rooms is a wall, which is represented analogously by an electrical circuit with three resistors and two capacitors known as 3R2C model in the literature (Xu and Wang 2007). The corresponding graph for this example is shown in Fig. Figure 3. In developing a model for this system, the sum of the energy losses through all pathways connected to a node, resulting in:

$$C_i^r \frac{dT_i}{dt} = Q_i^{in} - \sum_{j \in \mathcal{N}_i} \sum_{k_j \in \mathcal{P}_j} q_{ij}^{k_j}, \tag{4}$$

$$\begin{bmatrix} q_{ij} \\ q_{ji} \end{bmatrix} = \frac{1}{B_{ij}(s)} \begin{bmatrix} A_{ij}(s) & -D_{ij}(s) \\ -D_{ij}(s) & A_{ji}(s) \end{bmatrix} \begin{bmatrix} T_i \\ T_j \end{bmatrix} \tag{5}$$

where $A_{ij}(s)$, $A_{ji}(s)$, $B_{ij}(s)$, and $D_{ij}(s)$ are defined in (Moore et al. 2011). $\mathcal{N}_i$ is the set of neighbors to which a node $i$ is connected and $\mathcal{P}_j$ is the set of pathways $k_j$ associated with any neighbor $j$ of node $i$. We note that $q_{ij} = \sum_{k_j \in \mathcal{P}_j} q_{ij}^{k_j}$. The parameter $C_i^r$ is the thermal capacity (mass) of the room $i$.

Combining (4) and (5) for the configuration shown in Fig. Figure 3 and defining the vectors

$$T(s) = \left[T_1(s)\ T_2(s)\ T_3(s)\ T_4(s)\right]^T, \quad Q^{in}(s) = \left[Q_1^{in}(s)\ Q_2^{in}(s)\ Q_3^{in}(s)\ Q_4^{in}(s)\right]^T,$$

we can easily show that:

$$sT(s) = Q^{in}(s) - L(s)T(s), \tag{6}$$

where the matrix $L(s) = [L_{ij}(s)]$ is given as:

$$L_{ij}(s) = \begin{cases} \sum_{j \in \mathcal{N}_i} \lambda_{ij}^S(s) & i = j \\ -\lambda_{ij}^C(s) & i \neq j \text{ and } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

We will refer to $L(s)$ defined in this way as a *dynamic Laplacian matrix*. For the graph topology shown in Fig. Figure 3, the dynamic Laplacian matrix has the form shown in (7). We have shown in (Moore et al. 2011) that when the weight matrices $\lambda_{ij}(s)$ satisfy certain assumptions, $-L(s)$ can be viewed as a *dynamic interconnection matrix*, allowing the demonstration of consensus.

Notice that we can redraw Fig. Figure 3 as shown in Fig. Figure 4, where

$$\lambda_{ij}(s) = \begin{bmatrix} \lambda_{ij}^S(s) & -\lambda_{ij}^C(s) \end{bmatrix} = \begin{bmatrix} \left(\frac{A_{ij}}{B_{ij}}\right)_w + \frac{1}{R_{ij}^d} & -\left(\frac{1}{B_{ij}}\right)_w - \frac{1}{R_{ij}^d} \end{bmatrix}, \tag{8}$$

The graph shown in Fig. Figure 4 will be referred-to as a *dynamic graph* or a *dynamic consensus network*. Then applying the definition of the dynamic Laplacian (7) for the dynamic graph Fig. Figure 4 we get

$$L(s) = \begin{bmatrix} \sum_{j=2,3,4} \lambda_{1j}^S(s) & -\lambda_{12}^C(s) & -\lambda_{13}^C(s) & -\lambda_{14}^C(s) \\ -\lambda_{21}^C(s) & \sum_{j=1,4} \lambda_{2j}^S(s) & 0 & -\lambda_{24}^C(s) \\ -\lambda_{31}^C(s) & 0 & \sum_{j=1,4} \lambda_{3j}^S(s) & -\lambda_{34}^C(s) \\ -\lambda_{41}^C(s) & -\lambda_{42}^C(s) & -\lambda_{43}^C(s) & \sum_{j=1,2,3} \lambda_{4j}^S(s) \end{bmatrix}, \tag{9}$$

which reduces to (7) if we insert the full expressions for $\lambda_{ij}^C(s)$ and $\lambda_{ij}^S(s)$ defined in (8). This leads us to consider the idea of *dynamic consensus networks*.

Recently, we consider another example motiviated a generalization of the static consensus problem (1), modeling of the load frequency control (LFC) network of an electrical power grid as dynamic consensus network (Oh et al. 2015). We consider the following network:

$$Y_i(s) = \frac{1}{s} \sum_{j \in \mathcal{N}_i} G_i(s) a_{ij} (Y_j(s) - Y_i(s)), \ i = 1, \ldots, N, \tag{10}$$

which can be viewed as a single-integrator consensus network with dynamic interconnection coefficients $G_i(s)a_{ij}$. In the LFC network of the grid, the output of each individual system is the phase of its voltage, which is the integration of the angular velocity. The interconnection is power exchanges among the individual systems through transmission lines, which are dependent on phase differences. Thus the LFC network has diffusive output interconnection. Further, individual systems have the phase difference through a transfer function $G_i(s)$, which includes the dynamics of their governor, turbine, generator, and local controller.

### 1.3 *More General Nodes Dynamics than a Single Integrators*

For the autonomous case ($Q_i^{in}(t) = 0$) and assume for the generalisty $x_i(t) = T_i(t)$, the dynamic of each node in (6) can then be written as:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} \lambda_{ij}(t) * [x_j(t) - x_i(t)]. \tag{11}$$

By analogy with the static case (1), (11) is referred to as the *dynamic consensus protocol.* Motivated by the single integrator dynamic consens network (11), we consider more complicated dynamics ($p_i(s)$ for the node $i$, $i = 1, 2, ..., N$) than simple integrator nodes as shown in Fig. Figure 5. For this dynamic network, we suppose that the node dynamic is given by

$$\dot{x}_i(t) = p_i(t) * \sum_{j \in \mathcal{N}_i} \lambda_{ij}(t) * [x_j(t) - x_i(t)]. \tag{12}$$

By taking the Laplace transform of both sides of (12), we get

$$x_i(s) = p_i(s) \sum_{j \in \mathcal{N}_i} \lambda_{ij}(s)[x_j(s) - x_i(s)]. \tag{13}$$

We call a dynamic network, such as that defined in (13), a dynamic consensus network with heterogeneous nodes and dynamic edges. If we consider identical LTI nodes, i.e., $p_i(s) = p(s)$, we can rewrite (13) as

$$x_i(s) = p(s) \sum_{j \in \mathcal{N}_i} \lambda_{ij}(s)[x_j(s) - x_i(s)]. \tag{14}$$

We call this protocol a dynamic consensus protocol with homogeneous nodes and dynamic edges.

Based on the dynamics of a network's nodes and their topology, several consensus problems can be specified as shown in Table 1. Consensus problems 1 and 2 have been the subject of many research works, whereas consensus problems 3 and 4 have not.

1: Network Structures and Corresponding Consensus Problems

| Case | Nodes | Arcs (Edges) | Consensus Problem |
|---|---|---|---|
| 1 | no processing | static weighted edges | weighted Laplacian |
| 2 | integrating nodes | static weighted edges | static consensus |
| 3 | integrating nodes | dynamic edges | dynamic consensus |
| 4 | dynamic nodes | dynamic edges | general dynamic consensus |

This paper focuses on the third and fourth cases: a network consisting of integrating nodes and a network composed of more general dynamic nodes connected by dynamic edges. We can consider two types of dynamic consensus networks: directed and undirected.

For dynamic consensus network as in Fig. Figure 4, less concern has been made towards controller design under external disturbances. The presence of disturbances might lead to oscillation or divergence of consensus in the proposed dynamic networks. It is of significance to investigate the effects of disturbances on the behavior of dynamic consensus networks and propose an appropriate controller to make dynamic consensus networks robust to disturbances and forcing node variables to converge to consensus when disturbances $w(t) \equiv 0$. Note that for the network shown in Fig. Figure 4, we assume the disturbances are zeros, whereas later will add disturbances to the plant network. In this paper, we propose to study problems related to disturbance

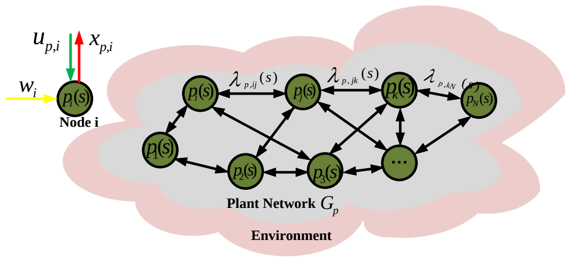


Figure 5: Plant as a dynamic network embedded in its environment.

attenuation within undirected dynamic consensus networks of identical linear systems that are under exogenous disturbances. For the proposed dynamic consensus networks, we will propose a methodology to formulate $H_\infty$ suboptimal problems based on decentralized controller.

The paper is organized as follow: Section 2 presents the modeling of a dynamic network viewed as the "plant" in an environment and we also propose a decentralized controller for minimizing the effect of disturbances on the disagreement vector and then we describe the performances criterion and a controller design methodology for disturbance rejection. In Section 3, we solve the $H_\infty$ suboptimal control problem of minimizing the effect of the disturbance on the disagreement vector. Finally, we give an example to illustrate our result. Lastly, we give conclusions and future works.

## 2 Problem Formulation

This section presents the modeling of a dynamic network viewed as the "plant" in an environment that includes inputs, outputs, and disturbances. We also propose a decentralized controller for minimizing the effect of disturbances on the disagreement vector.

### 2.1 *The Plant Network and its Environment*

Consider the plant network $G_p \in (\mathcal{V}, \mathcal{E}_p, \lambda_{p,ij}(s))$ shown in Fig. Figure 5. Assume this network consists of $N$ identical linear systems ($p_i(s) = p(s), \forall i \in 1, 2, \ldots, N$) interconnected by a set of edges $\mathcal{E}_p(s) = \{(n_i, n_j) : n_i, n_j \in \mathcal{N}_{p,i}\}$. We assume the nodes of the plant network have state $x_i$ (denoted by red lines), input $u_i$ (denoted by green lines), and disturbance $w_i$ (denoted by yellow lines). Later we also introduce outputs for each node in the plant network $G_p$. Here we assume all nodes are identical and all edges are identical. In this case we see that it is possible to use a decomposition procedure to design a controller. To accomplish this we write a state-space representation of the plant network in a way that can be suitable for the decomposition procedure.

Let the dynamics of the nodes in the given plant network $G_p$ with external disturbances $w_i$ be modeled by the following state-space equations:

$$\begin{aligned}\dot{x}_{p,i}(t) &= A_p x_{p,i}(t) + B_p u_{p,i}(t) + F w_i(t)\\ y_{p,i}(t) &= C_p x_{p,i}(t), \forall i = 1 \ldots N,\end{aligned} \tag{15}$$

where $x_{p,i}(t) \in \mathbb{R}^n$, $u_{p,i}(t) \in \mathbb{R}^{m_p}$, $w_i(t) \in \mathbb{R}^q$, $y_{p,i}(t) \in \mathbb{R}^p$ denote the state, input, external disturbance, and output, respectively, of node $i$ for $i = 1, \ldots, N$, and $A_p, B_p, C_p$ are constant matrices with appropriate dimensions. If we define the vectors $x_p(t) = [x_{p,1}(t)^T, ..., x_{p,N}(t)^T]^T$, $u_p(t) = [u_{p,1}(t)^T, ..., u_{p,N}(t)^T]^T$, $w(t) = [w_1(t)^T, ..., w_N(t)^T]^T$, and

$y_p(t) = [y_{p,1}(t)^T, ..., y_{p,N}(t)^T]^T$, we can write (15) in matrix form as:

$$\begin{aligned}\dot{x}_p(t) &= (I_N \otimes A_p)x_p + (I_N \otimes B_p)u_p(t) + (I_N \otimes F)w(t)\\ y_p(t) &= (I_N \otimes C_p)x_p(t).\end{aligned} \tag{16}$$

Note that the input vector into the nodes $u_p(t)$ in the above equation is given by:

$$u_p(t) = u^{in}(t) + u^e(t), \tag{17}$$

where $u^{in}(t) = [u_1^{in}(t)^T, ..., u_N^{in}(t)^T]^T$ is the input vector from the environment to the nodes in $G_p$ and $u^e(t) = [u_1^e(t)^T, \ldots, u_N^e(t)^T]^T$ is the input vector from the dynamic topology (edges) to the nodes. Next, suppose the interconnections (edges) of the plant graph have identical LTI dynamics (transfer functions $e_{ij}(s) = H(s)$, for all $i$, $j$). Thus the dynamic consensus protocol that describes the dynamic topology (the dynamics of the edges) is given by the following equations:

$$u_i^e(s) = \sum_{j\in\mathcal{N}_{p,i}} [H(s)(y_{p,i}(s) - y_{p,j}(s))] = H(s) \sum_{j\in\mathcal{N}_{p,i}} [(y_{p,i}(s) - y_{p,j}(s))], \forall i = 1 \ldots N, \tag{18}$$

where $\mathcal{N}_{p,i}$ is the neighborhood of node $i$ in the plant network $G_p$. If the identical dynamics of the edges $H(s)$ have a state-space realization $(A_e, B_e, C_e, D_e)$, the model (18) can be written in time domain as:

$$\begin{aligned}\dot{x}_{e,i}(t) &= A_e x_{e,i}(t) + B_e \sum_{j\in\mathcal{N}_{p,i}} (y_{p,i}(t) - y_{p,j}(t))\\ u_i^e(t) &= C_e x_{e,i}(t) + D_e \sum_{j\in\mathcal{N}_{p,i}} (y_{p,i}(t) - y_{p,j}(t)), \forall i = 1, \ldots, N,\end{aligned} \tag{19}$$

where $x_{e,i}(t) \in \mathbb{R}^{\bar{n}}$ denotes the state of identical edges. If we define the vectors $x_e(t) = [x_{e,1}(t)^T, \ldots, x_{e,N}(t)^T]^T$, we can write the dynamics of the overall system of the dynamic consensus protocol (edges model, (19)) as:

$$\begin{aligned}\dot{x}_e(t) &= (I_N \otimes A_e)x_e(t) + (L_p \otimes B_e)y_p(t)\\ u^e(t) &= (I_N \otimes C_e)x_e(t) + (L_p \otimes D_e)y_p(t),\end{aligned} \tag{20}$$

where $L_p$ is unweighted, static Laplacian matrix of the plant network $G_p$.

Combining (16), (17), and (20), the plant network $G_p$ can be modeled in a matrix form as:

$$\begin{bmatrix}\dot{x}_p(t)\\ \dot{x}_e(t)\end{bmatrix} = \begin{bmatrix} I_N \otimes A_p + L_p \otimes B_p D_e C_p & I_N \otimes B_p C_e\\ L_p \otimes B_e C_p & I_N \otimes A_e\end{bmatrix} \begin{bmatrix}x_p(t)\\ x_e(t)\end{bmatrix} + \begin{bmatrix}I_N \otimes B_p\\ \mathbf{0}\end{bmatrix} u^{in}(t) + \begin{bmatrix}I_N \otimes F\\ \mathbf{0}\end{bmatrix} w(t),$$

where $(A_p, B_p, C_p)$ and $(A_e, B_e, C_e, D_e)$ are the state space realization of the nodes and edges of the plant network $G_p$, respectively, and $B_p$, $F$ are the input and disturbance matrices.

### 2.2 *Decentralized Controller*

Using the neighbor's measured state $x_{p,i}$, we propose the following decentralized controller:

$$u_i^{in}(t) = k_c K \left(\textstyle\sum_{j\in\mathcal{N}_{p,i}} [w_{ij}(x_{p,i}(t) - x_{p,j}(t))]\right), \forall i = 1, 2, ..., N, \tag{21}$$

where $k_c > 0$ is a constant (scalar), $K \in \mathbb{R}^{m\times n}$ is a static gain matrix, $\mathcal{N}_{p,i}$ is the neighborhood of node $i$ in the plant network $G_p$, and $A = (w_{ij})_{N\times N}$ is the unweighted, static adjacency matrix of the plant network $G_p$. Note that $k_c$ and $K$ are design parameters (to be determined) for the proposed decentralized controller (21). Although we could combine $k_cK$ into a signal term, it is convenient in our later analysis to view them separately.

In matrix form, (21) can be written as

$$u^{in}(t) = k_c(L_p \otimes K)x_p(t), \tag{22}$$

where $L_p$ is the unweighted, static Laplacian matrix of the plant network, $k_c > 0$ is a constant (scalar), and $K \in \mathbb{R}^{m\times n}$ is a static gain matrix.

### 2.3 *Performances Criterion, and the Closed-Loop System*

Using the neighbor's measured state $x_{p,i}$, assume the plant network $G_p$ is controlled using the proposed decentralized controller, which is defined in (22). We say that a decentralized controller (22) asymptotically solves the dynamic consensus problem if all the node variables in the plant network asymptotically reach consensus. In other words, if and only if the states of the nodes of the plant network satisfy

$$\lim_{t\to\infty} (x_{p,i}(t) - x_{p,j}(t)) = 0 \ \ \forall i, j \in \{1, 2, \cdots N\}. \tag{23}$$

The objective in this section is to design a decentralized controller to attenuate the influence of external disturbances when $w \not\equiv 0$ and make all states of the nodes of the plant network (23) achieve agreement or consensus when $w \equiv 0$. Following (Lin and Jia 2010), a natural way to analyze the effect of the external disturbances on consensus is to define a controlled-output function $z_i(t)$ as:

$$z_i(t) = \frac{1}{N}\sum_{j=1}^{N}[x_{p,i}(t) - x_{p,j}(t)] = x_{p,i}(t) - \frac{1}{N}\sum_{j=1}^{N} x_{p,j}(t), \ \ \forall i = 1, ..., N. \tag{24}$$

We can note from the above equation that $z_i(t)$ is a measure of the disagreement of state $x_i(t)$ relative to the average state of all nodes. If we define $z(t) = [z_1(t)^T, \cdots, z_N(t)^T]^T$, then we can write the disagreement equation (24) in a matrix form as:

$$z(t) = (L_g \otimes I_n)x_p(t), \tag{25}$$

where $L_g = [l_{ij}]$ is the disagreement matrix, which is defined as:

$$l_{ij} = \begin{cases} \frac{N-1}{N} & i = j \\ -\frac{1}{N} & i \neq j \end{cases}. \tag{26}$$

$z(t)$ is called the disagreement vector. $L_g$ is the static Laplacian of a complete graph, multiplied by $1/N$. Based on (24) and (26), it can be seen that $\mathbf{1}_N^T L_g = \mathbf{0}_N^T$, $L_g\mathbf{1}_N = \mathbf{0}_N$, $L_g^2 = L_g$, and that we can write the disagreement matrix $L_g$ as:

$$L_g = I_N - \frac{\mathbf{1}_N\mathbf{1}_N^T}{N}. \tag{27}$$

Combining (25) and (27), we get

$$z(t) = x_p(t) - \frac{\mathbf{1}_{nN}\mathbf{1}_{nN}^T}{N} x_p(t). \tag{28}$$

By combining (23) and (25), the closed-loop system of the dynamic network, including the decentralized controller (22), can be written as

$$\begin{aligned}
\begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_e(t) \end{bmatrix} &= \begin{bmatrix} I_N \otimes A_p + L_p \otimes B_p D_e C_p + k_c(L_p \otimes B_p K) & I_N \otimes B_p C_e \\ L_p \otimes B_e C_p & I_N \otimes A_e \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_e(t) \end{bmatrix} + \begin{bmatrix} I_N \otimes F \\ \mathbf{0} \end{bmatrix} w(t) \\
&\triangleq A_{cl} \begin{bmatrix} x_p(t) \\ x_e(t) \end{bmatrix} + B_{cl} w(t) \\
z(t) &= \begin{bmatrix} L_g \otimes I_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_e(t) \end{bmatrix} \triangleq C_{cl} \begin{bmatrix} x_p(t) \\ x_e(t) \end{bmatrix},
\end{aligned} \tag{29}$$

where $L_p$ in (29) is the unweighted, static Laplacian matrix of the plant network $G_p$ and $L_g$ is the disagreement Laplacian matrix.

## 3 $H_\infty$ Suboptimal Control Problem

In this section we solve the problem of minimizing the effect of the disturbance on the disagreement vector. This problem has been considered in (Oh 2013) for the case where the plant has static weights between nodes. The solutions in (Oh 2013) were strongly motivated by the analysis paper in (Li et al. 2011), which showed how to compute $H_\infty$ and $H_2$ performance in networks with static weights as a function of those weights. The innovation in (Oh 2013) was to pose a control problem and then apply the results of (Li et al. 2011) to pick control parameters and plant weights to minimize $H_\infty$ and $H_2$ performance. The innovation in this section is to extend the results of (Oh 2013) to the dynamic weight case, again exploiting the computational results in (Li et al. 2011). In particular, equation (29) is novel to our work and the proofs from (Oh 2013) and (Li et al. 2011), though exploited, are modified to consider the addition of terms in equation (29) that account for the effect of the dynamic edges (e.g., terms associated with $x_e(t)$).

As was mentioned earlier, the disagreement vector $z(t)$ can bee seen as a measure of the consensus of the closed loop system (29). One way to measure the ability of the closed loop system to achieve consensus versus disturbance attenuation is using the $H_\infty$ norm of the closed-loop transfer function matrix from $w$ to $z$. Thus the controller design problem can be restated as the following $H_\infty$ suboptimal control problem:

Problem 3.1 Given a plant network $G_p$ with identical LTI nodes and dynamic edges, for a given $\gamma > 0$, design a decentralized controller (22) (design the scalar $k_c$ and the static gain matrix $K$) such that the closed loop system (29) asymptotically achieves consensus when $w(t) \equiv 0$ and $\|T_{zw}(\jmath\omega)\|_\infty < \gamma$ when $w(t) \not\equiv 0$.

Most results in the recent literature have been focused on controller design to ensure disturbance reduction for static networks with the edges modeled as positive gains with possible communication time-delays. Here we extend these results for the case of identical dynamic edges. To proceed, note that the properties of the Laplacian matrix of the plant network $L_p$ and the disagreement Laplacian matrix $L_g$ allow the closed-loop system to be decomposed into a set of smaller subsystems. This, together with results from the theory of linear matrix inequalities (LMIs), will allow the development of a procedure for controller design that meets a prescribed $H_\infty$ performance criteria. For more details regarding this problem for the static case, see (Li et al. 2012, Lin and Jia 2008). In (Lin et al. 2008) derived conditions to guarantee that all agents reach consensus while satisfying the desired $H_\infty$ performance in fixed and switching topologies of directed networks with external disturbances and model uncertainty in both the absence and presence of time-delays. In order to derive these conditions, they performed a model transformation to transform the original system into a reduced-order system because the closed loop system matrix $-L$ is singular, and then the traditional $H_\infty$ control theory is invalid.

Here we use three transformations to decompose the closed loop system (29) into $N-1$ lower-order systems. This reduces the complexity of computation relative to the full $N \times N$ multi-variable system. For clarity, we do these transformations separately in four steps:

### 3.1 *Step 1: Model Transformation*

The model transformation is given by the following:

$$\hat{x}_p(t) = [L_g \otimes I_n]x_p(t), \quad \hat{x}_e(t) = [L_g \otimes I_{\bar{n}}]x_e(t) \tag{30}$$

Note that the transformation (30) is not invertible. However, the non-zero eigenvalues in the system are preserved, so that the non-zero eigenvalues of (29) and the non-zeros eigenvalues of the transformed system (32) below are the same.

From (28) we have

$$x_p(t) = \hat{x}_p(t) + \frac{\mathbf{1}_{nN}\mathbf{1}_{nN}^T}{N}x_p(t), \quad x_e(t) = \hat{x}_e(t) + \frac{\mathbf{1}_{nN}\mathbf{1}_{nN}^T}{N}x_e(t) \tag{31}$$

By performing the model transformation (30) on the closed loop system (29) and using (31), and the properties of the Kronecker product $((A_1 \otimes A_2)(B_1 \otimes B_2) = (A_1B_1 \otimes A_2B_2))$, we get the following equivalent closed loop system:

$$\begin{aligned}
\begin{bmatrix}\dot{\hat{x}}_p(t)\\ \dot{\hat{x}}_e(t)\end{bmatrix} &= \begin{bmatrix} L_g \otimes A_p + L_gL_p \otimes B_pD_eC_p + k_c(L_gL_p \otimes B_pK) & L_g \otimes B_pC_e \\ L_gL_p \otimes B_eC_p & L_g \otimes A_e\end{bmatrix}\begin{bmatrix}\hat{x}_p(t)\\ \hat{x}_e(t)\end{bmatrix} + \begin{bmatrix} L_g \otimes F \\ \mathbf{0}\end{bmatrix} w(t) \\
&\triangleq \hat{A}_{cl}\begin{bmatrix}\hat{x}_p(t)\\ \hat{x}_e(t)\end{bmatrix} + \hat{B}_{cl}w(t) \\
z(t) &= \begin{bmatrix} L_g \otimes I_n & \mathbf{0}\end{bmatrix}\begin{bmatrix}\hat{x}_p(t)\\ \hat{x}_e(t)\end{bmatrix} \triangleq \hat{C}_{cl}\begin{bmatrix}\hat{x}_p(t)\\ \hat{x}_e(t)\end{bmatrix}.
\end{aligned} \tag{32}$$

where this result is due to the fact $L_g\mathbf{1}_N = \mathbf{0}_N$, $L_c\mathbf{1}_N = \mathbf{0}_N$, and then $[L_g \otimes A_p + L_gL_p \otimes B_pD_eC_p + k_c(L_gL_p \otimes B_pK)]\mathbf{1}_{nN} = \mathbf{0}_{nN}$ and $[L_g \otimes B_pC_e]\mathbf{1}_{nN} = \mathbf{0}_{nN}$.

### 3.2 *Step 2: Orthogonal Transformation*

From the definition of the static Laplacian matrix and the definition of the disagreement matrix (26), we can summarize the properties of $L_g$ and $L_p$ in the following lemma:

**Lemma 3.2:** *(Lin et al. 2008) Let $L_g \in \mathbb{R}^{N\times N}$ and $L_p \in \mathbb{R}^{N\times N}$ be Laplacian matrices associated with connected graphs with $L_g$ the complete graph. Then the following statements hold:*

1. *The eigenvalues of $L_g$ are $\mathbf{1}_N$ with multiplicity $N-1$ and 0 with multiplicity 1. The vectors $\mathbf{1}_N^T$ and $\mathbf{1}_N$ are the left and right eigenvectors of $L_g$ associated with the zero eigenvalue, respectively.*
2. *There exists an orthogonal matrix $Q \in \mathbb{R}^{N\times N}$ such that:*

$$Q^TL_gQ = \begin{bmatrix} 0 & 0 \\ \mathbf{0}_{N-1} & I_{N-1}\end{bmatrix} = \overline{L}_g, \quad Q^TL_pQ = \begin{bmatrix} 0 & 0 \\ \mathbf{0}_{N-1} & \widetilde{L}_p\end{bmatrix} = \overline{L}_p, \tag{33}$$

*and the first column of $Q$ is $\frac{\mathbf{1}_N}{\sqrt{N}}$, where $\widetilde{L}_p \in \mathbb{R}^{N-1\times N-1}$ is positive definite matrix if and only if the plant network is connected.*

By (33), the orthogonal transformation for the closed loop system (32) can be accomplished using the following orthogonal transformations:

$$\begin{aligned}\bar{x}_p(t) &= (Q^T \otimes I_n)\hat{x}_p(t), \bar{x}_e(t) = (Q^T \otimes I_{\bar{n}})\hat{x}_e(t),\\ \bar{w}(t) &= (Q^T \otimes I_q)w(t), \bar{z}(t) = (Q^T \otimes I_n)z(t).\end{aligned} \tag{34}$$

By performing the orthogonal transformation (34) on the closed loop system (29) and using (33), the equivalent closed loop system can be written as

$$\begin{aligned}\begin{bmatrix}\dot{\bar{x}}_p(t)\\ \dot{\bar{x}}_e(t)\end{bmatrix} &= \begin{bmatrix}\overline{L}_g \otimes A_p + \overline{L}_g\overline{L}_p \otimes B_pD_eC_p + k_c(\overline{L}_g\overline{L}_p \otimes B_pK) & \overline{L}_g \otimes B_pC_e\\ \overline{L}_g\overline{L}_p \otimes B_eC_p & \overline{L}_g \otimes A_e\end{bmatrix}\begin{bmatrix}\bar{x}_p(t)\\ \bar{x}_e(t)\end{bmatrix} + \begin{bmatrix}\overline{L}_g \otimes F\\ \mathbf{0}\end{bmatrix}\bar{w}(t)\\ &\triangleq \bar{A}_{cl}\begin{bmatrix}\bar{x}_p(t)\\ \bar{x}_e(t)\end{bmatrix} + \bar{B}_{cl}\bar{w}(t)\\ \bar{z}(t) &= \begin{bmatrix}\overline{L}_g \otimes I_n & \mathbf{0}\end{bmatrix}\begin{bmatrix}\bar{x}_p(t)\\ \bar{x}_e(t)\end{bmatrix} \triangleq \bar{C}_{cl}\begin{bmatrix}\bar{x}_p(t)\\ \bar{x}_e(t)\end{bmatrix}.\end{aligned} \tag{35}$$

### 3.3 *Step 3: Model Reduction*

From (33), we get

$$\overline{L}_g\overline{L}_p = \begin{bmatrix}0 & 0\\ \mathbf{0}_{N-1} & I_{N-1}\end{bmatrix}\begin{bmatrix}0 & 0\\ \mathbf{0}_{N-1} & \widetilde{L}_p\end{bmatrix} = \begin{bmatrix}0 & 0\\ \mathbf{0}_{N-1} & \widetilde{L}_p\end{bmatrix}. \tag{36}$$

From (36), we can note that the first rows of the matrices $\overline{L}_g$, $\overline{L}_p$ and $\overline{L}_g\overline{L}_p$ are all zeros. Thus, we can eliminate them and the closed loop system (35) can be written as

$$\begin{aligned}\begin{bmatrix}\dot{\widetilde{x}}_p(t)\\ \dot{\widetilde{x}}_e(t)\end{bmatrix} &= \begin{bmatrix}I_{N-1} \otimes A_p + \widetilde{L}_p \otimes B_pD_eC_p + k_c(\widetilde{L}_p \otimes B_pK) & I_{N-1} \otimes B_pC_e\\ \widetilde{L}_p \otimes B_eC_p & I_{N-1} \otimes A_e\end{bmatrix}\begin{bmatrix}\widetilde{x}_p(t)\\ \widetilde{x}_e(t)\end{bmatrix} + \begin{bmatrix}I_{N-1} \otimes F\\ \mathbf{0}\end{bmatrix}\widetilde{w}(t)\\ &\triangleq \widetilde{A}_{cl}\begin{bmatrix}\widetilde{x}_p(t)\\ \widetilde{x}_e(t)\end{bmatrix} + \widetilde{B}_{cl}\widetilde{w}(t)\\ z(t) &= \begin{bmatrix}I_{N-1} \otimes I_n & \mathbf{0}\end{bmatrix}\begin{bmatrix}\widetilde{x}_p(t)\\ \widetilde{x}_e(t)\end{bmatrix} \triangleq \widetilde{C}_{cl}\begin{bmatrix}\widetilde{x}_p(t)\\ \widetilde{x}_e(t)\end{bmatrix},\end{aligned} \tag{37}$$

where $\widetilde{x}_p(t) = [\bar{x}_{p,2}(t)^T, \ldots, \bar{x}_{p,N}(t)^T]^T$, $\bar{x}_e(t) = [\bar{x}_{e,2}(t)^T, \ldots, \bar{x}_{e,N}(t)^T]^T$, $\bar{w}(t) = [\bar{w}_N(t)^T, \ldots, \bar{w}_2(t)^T]^T$, and $\bar{z}(t) = [\bar{z}_2(t)^T, \ldots, \bar{z}_N(t)^T]^T$. Note that technically this step is not a transformation, but is more like a projection, as we are relating only the non-zero eigenvalues of the original system.

### 3.4 *Step 4: Diagonalization*

Now we introduce a diagonal transformation procedure. For a connected plant network, the matrix $\widetilde{L}_p \in \mathbb{R}^{N-1\times N-1}$ is a symmetric and positive definite matrix and then there exist an orthogonal matrix $U$ such that

$$U^T\widetilde{L}_pU = \Lambda_p = diag(\lambda_{p,2}, \lambda_{p,3}, ..., \lambda_{p,N}), \tag{38}$$

where $\lambda_{p,i}$, $\forall i = 2 \dots N$, are the $i^{th}$ eigenvalues of $\widetilde{L}_p$. Note that the eigenvalues of $L_p$ are always positive because of the positivity definiteness of this matrix when the associated graph is connected. Thus define

$$\begin{aligned} \grave{x}_p(t) &= (U^T \otimes I_n)\widetilde{x}_p(t), \grave{x}_e(t) = (U^T \otimes I_{\bar{n}})\widetilde{x}_e(t), \\ \grave{w}(t) &= (U^T \otimes I_q)\widetilde{w}(t), \grave{z}(t) = (U^T \otimes I_n)\widetilde{z}(t). \end{aligned} \tag{39}$$

Using the diagonal transformations (39), the decomposed system (37) can be written as

$$\begin{aligned} \begin{bmatrix} \dot{\grave{x}}_p(t) \\ \dot{\grave{x}}_e(t) \end{bmatrix} &= \begin{bmatrix} I_{N-1} \otimes A_p + \Lambda_p \otimes B_p D_e C_p + k_c(\Lambda_p \otimes B_p K) & I_{N-1} \otimes B_p C_e \\ \Lambda_p \otimes B_e C_p & I_{N-1} \otimes A_e \end{bmatrix} \begin{bmatrix} \grave{x}_p(t) \\ \grave{x}_e(t) \end{bmatrix} + \begin{bmatrix} I_{N-1} \otimes F \\ \mathbf{0} \end{bmatrix} \grave{w}(t) \\ &\triangleq \grave{A}_{cl} \begin{bmatrix} \grave{x}_p(t) \\ \grave{x}_e(t) \end{bmatrix} + \grave{B}_{cl}\grave{w}(t) \\ \grave{z}(t) &= \begin{bmatrix} I_{N-1} \otimes I_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} \grave{x}_p(t) \\ \grave{x}_e(t) \end{bmatrix} \triangleq \grave{C}_{cl} \begin{bmatrix} \grave{x}_p(t) \\ \grave{x}_e(t) \end{bmatrix} \end{aligned} \tag{40}$$

We have shown that the original closed loop system (29) can be expressed by the diagonal closed loop system (40). To proceed we will design the decentralized controller (22) for the decomposed system. This is possible because it can be shown (below) that $\|T_{zw}(s)\|_\infty$ of the original system (29) is the same as $\|T_{\grave{z}\grave{w}}(s)\|_\infty$ of the diagonal system (40). This is true even through the first transformation (30) is not invertible, because as described above, all the systems have the same non-zero eigenvalues. Next section presents the main result.

## 4 Main Result

Based on Bounded Real Lemma in (Boyd et al. 1994) motivated by Theorem 3 in (Li et al. 2011), Lemma 9.4.2 and derivative ideas in (Oh 2013), we have the following Lemma:

**Lemma 4.1:** *Consider the dynamic consensus network $G_p$ controlled by a decentralized controller network defined in (22). Assume the dynamic network is connected. For a given prescribed value of the attenuation performance $\gamma > 0$, the reduced order system (40) asymptotically achieves consensus when $w(t) \equiv 0$ and satisfies $\|T_{\grave{z}\grave{w}}(s)\|_\infty = \|T_{zw}(s)\|_\infty < \gamma$ when $w(t) \not\equiv 0$, if there exist a positive definite matrices $P_i$, $i = 2, \dots, N$, such that the following Riccati matrix inequalities are satisfied:*

$$\grave{A}_i^T P_i + P_i \grave{A}_i + \frac{1}{\gamma^2} P_i \grave{F}_0 \grave{F}_0^T P_i + \grave{C}_0^T \grave{C}_0 < 0, \forall i = 2, \dots, N, \tag{41}$$

*where*

$$\begin{aligned} \grave{A}_i &= \grave{A}_i^0 + k_c \lambda_{p,i} \grave{B}_0 \grave{K}_0, \\ \grave{A}_i^0 &= \begin{bmatrix} A_p + \lambda_{p,i} B_p D_e C_p & B_p C_e \\ \lambda_{p,i} B_e C_p & A_e \end{bmatrix}, \grave{B}_0 = \begin{bmatrix} B_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ \grave{K}_0 &= \begin{bmatrix} K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \grave{F}_0 = \begin{bmatrix} F \\ \mathbf{0} \end{bmatrix}, \grave{C}_0 = \begin{bmatrix} I_n & \mathbf{0} \end{bmatrix}, \end{aligned} \tag{42}$$

*where $\lambda_{p,i}$ is the $i^{th}$ non-zero eigenvalues of the Laplacian matrix $L_p$.*

**Proof:**

To begin, we note that because of the diagonal block structure of the system matrix in (40), by using (38) and rearranging the states of the decomposed system (40), we get

$$\begin{aligned}\begin{bmatrix}\dot{\grave{x}}_{p,i}(t)\\ \dot{\grave{x}}_{e,i}(t)\end{bmatrix} &= \begin{bmatrix}A_p+\lambda_{p,i}B_pD_eC_p+k_c\lambda_{p,i}B_pK & B_pC_e\\ \lambda_{p,i}B_eC_p & A_e\end{bmatrix}\begin{bmatrix}\grave{x}_{p,i}(t)\\ \grave{x}_{e,i}(t)\end{bmatrix}+\begin{bmatrix}F\\ \mathbf{0}\end{bmatrix}\grave{w}_i(t)\\ &\triangleq \grave{A}_i\begin{bmatrix}\grave{x}_{p,i}(t)\\ \grave{x}_{e,i}(t)\end{bmatrix}+\grave{F}_0\grave{w}_i(t)\\ \grave{z}_i(t) &= \begin{bmatrix} I_n & \mathbf{0}\end{bmatrix}\begin{bmatrix}\grave{x}_{p,i}(t)\\ \grave{x}_{e,i}(t)\end{bmatrix}\triangleq \grave{C}_0\begin{bmatrix}\grave{x}_{p,i}(t)\\ \grave{x}_{e,i}(t)\end{bmatrix}, \forall i = 2\dots N,\end{aligned} \tag{43}$$

or equivalently,

$$\begin{aligned}\dot{\grave{x}}_i(t) &= \underbrace{(\grave{A}_i^0 + k_c\lambda_{p,i}\grave{B}_0\grave{K}_0)}_{\grave{A}_i}\grave{x}_i(t) + \grave{F}_0\grave{w}_i(t)\\ \grave{z}_i(t) &= \grave{C}_0\grave{x}_i(t), \forall i = 2\dots N,\end{aligned} \tag{44}$$

where $\grave{x}_i(t) = [\grave{x}_{p,i}(t), \grave{x}_{e,i}(t)]^T$, $\lambda_{p,i}$ is the $(i)^{th}$ non-zero eigenvalues of the Laplacian matrices $L_p$ of the plant network, and the system matrix $\grave{A}_i$, and $(\grave{F}_0, \grave{C}_0)$ can be decomposed and given as in (42).

We next need to show that the $H_\infty$ norm of the following transfer function matrices are the same:

$$\begin{aligned}&T_{zw}(s) = C_{cl}(sI - A_{cl})^{-1}B_{cl},\ \ T_{\bar{z}\bar{w}}(s) = \bar{C}_{cl}(sI - \bar{A}_{cl})^{-1}\bar{B}_{cl},\\ &T_{\widetilde{z}\widetilde{w}}(s) = \widetilde{C}_{cl}(sI - \widetilde{A}_{cl})^{-1}\widetilde{B}_{cl},\ T_{\grave{z}\grave{w}}(s) = \grave{C}_{cl}(sI - \grave{A}_{cl})^{-1}\grave{B}_{cl}.\end{aligned} \tag{45}$$

By using the model, orthogonal, and diagonal transformations, we can verify that $\|T_{zw}(s)\|_\infty = \|T_{\bar{z}\bar{w}}(s)\|_\infty = \|T_{\widetilde{z}\widetilde{w}}(s)\|_\infty = \|T_{\grave{z}\grave{w}}(s)\|_\infty$ as follows:
From (34) and (39) we have

$$\begin{aligned}&\bar{w}(t) = (Q^T \otimes I_q)w(t), \bar{z}(t) = (Q^T \otimes I_n)z(t),\\ &\grave{w}(t) = (U^T \otimes I_q)\widetilde{w}(t), \grave{z}(t) = (U^T \otimes I_n)\widetilde{z}(t).\end{aligned} \tag{46}$$

Using (46) we can write $T_{zw}(s)$ as

$$T_{zw}(s) = (Q^T \otimes I_n)T_{\bar{z}\bar{w}}(s)(Q \otimes I_q) = T_{\widetilde{z}\widetilde{w}}(s) = (U^T \otimes I_n)T_{\grave{z}\grave{w}}(s)(U \otimes I_q). \tag{47}$$

By observation $(Q^T \otimes I_n)$, $(Q \otimes I_q)$, $(U^T \otimes I_n)$, and $(U \otimes I_q)$ are unitary matrices, which leads to $\|T_{zw}(s)\|_\infty = \|T_{\bar{z}\bar{w}}(s)\|_\infty = \|T_{\widetilde{z}\widetilde{w}}(s)\|_\infty = \|T_{\grave{z}\grave{w}}(s)\|_\infty$.

Based on the definition of the $H_\infty$ norm, we can write

$$\|T_{zw}(s)\|_\infty = \|T_{\grave{z}\grave{w}}(s)\|_\infty = \max_{i=2,\dots,N}\|T_{\grave{z}_i\grave{w}_i}(s)\|_\infty. \tag{48}$$

Thus, from the above equation we conclude that the $H_\infty$ performance of the original system (29) can be investigated using the $H_\infty$ performance of $(N-1)$ reduced order systems (44).

Applying Bounded Real Lemma in (Boyd et al. 1994) on the system (44), we have that the system (44) with matrix transfer function $T_{\grave{z}_i\grave{w}_i}(s) = \grave{C}_{cl}(sI - \grave{A}_{cl})^{-1}\grave{B}_{cl}, \forall i = 2,\dots,N$ is asymptotically stable and $\|T_{zw}(s)\|_\infty = \|T_{\grave{z}\grave{w}}(s)\|_\infty < \gamma$, if there exist positive definite matrices

$P_i > 0, \forall i = 2, \ldots, N$, such that the following Riccati inequality is satisfied:

$$\grave{A}_i^T P_i + P_i \grave{A}_i + \frac{1}{\gamma^2} P_i \grave{F}_0 \grave{F}_0^T P_i + \grave{C}_0^T \grave{C}_0 < 0, \forall i = 2, \ldots, N, \tag{49}$$

Note that $\|T_{zw}(s)\|_\infty = \|T_{\grave{z}\grave{w}}(s)\|_\infty < \gamma$ follows from (48).

Since the system (44) is asymptotically stable (by Bounded Real Lemma (Boyd et al. 1994)), all eigenvalues (poles) of the system matrix $\grave{A}_i$ lie in the LHP or $\grave{A}_i$ is Hurwitz matrix, $\forall i = 2, \ldots, N$. Because we have assumed that the plant network including the controller network is connected, zero eigenvalue of the system matrix $A_{cl}$ is distinct (this is shown in (Moore et al. 2011)) and then all node variables of the plant network will come to consensus. □

Based on Lemma 4.1, we conclude that the $H_\infty$ suboptimal problem (Problem 3.1) of the original system (29) can be solved using the $(N-1)$ $H_\infty$ suboptimal problems of the decomposed system (44). The following theorem introduces a sufficient condition for existence of a decentralized controller (22) such that the original closed loop system (29) asymptotically achieves consensus when $w(t) \equiv 0$ and $\|T_{zw}(s)\|_\infty < \gamma$. This theorem is a standard LMI result for $H_\infty$-based design and is similar to Theorem 6 in (Li et al. 2011), which considers the static case, and Theorem 9.4.1 in (Oh 2013), which also considers the static case.

**Theorem 4.2:** *Consider the dynamic consensus network $G_p$ controlled by a decentralized controller network defined in (22). Assume the dynamic network is connected. For prescribed value of the attenuation performance $\gamma > 0$, the system (29) with the closed loop matrix transfer function $T_{zw}(s)$ asymptotically achieves consensus when $w(t) \equiv 0$ and $\|T_{zw}(s)\|_\infty < \gamma$ when $w(t) \not\equiv 0$, if there exist a positive definite matrix $Q = diag(Q_1, Q_2) = Q^T > 0$, and a scalar $\delta_{min} > 0$ such that the following LMIs are satisfied:*

$$\begin{bmatrix} Q(\grave{A}_i^0)^T + (\grave{A}_i^0)Q - \delta_{min}\grave{B}_0\grave{B}_0^T & \grave{F}_0 & Q\grave{C}_0^T \\ \grave{F}_0^T & -\gamma^2 I & \mathbf{0} \\ \grave{C}_0 Q & \mathbf{0} & -I \end{bmatrix} < 0, \quad \text{for } i = 2, N, \tag{50}$$

*where $\grave{A}_i^0$, $\grave{B}_0$, $\grave{C}_0$ and $\grave{F}_0$ are defined in (42). Furthermore, if the LMIs (50) hold, then the static gain matrix $K$ and the scalar $k_c$ of the proposed decentralized controller (22) are given by:*

$$\grave{K}_0 = \begin{bmatrix} K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = -\frac{1}{2}\grave{B}_0^T Q^{-1}, \text{ and } k_c \text{ chosen such that } k_c \geq \frac{\delta_{min}}{\lambda_{p,2}}. \tag{51}$$

*where $\lambda_{p,2}$ is the smallest non-zero eigenvalue of the Laplacian matrices $L_p$ of the plant network.*

**Proof:**

Suppose the inequality (50) is satisfied. Then, based on the Schur Complement Lemma in (Boyd et al. 1994), it follows from (50) that there exist a positive definite matrix $Q = diag(Q_1, Q_2) = Q^T > 0$ and a scalar $\delta_{min} > 0$ such that

$$Q(\grave{A}_i^0)^T + \grave{A}_i^0 Q - \delta_{min}\grave{B}_0\grave{B}_0^T + \frac{1}{\gamma^2}\grave{F}_0\grave{F}_0^T + Q\grave{C}_0^T\grave{C}_0 Q < 0, \text{for } i = 2, N. \tag{52}$$

Since $k_c \geq \frac{\delta_{min}}{\lambda_{p,2}}$ follows from (51), we have $k_c\lambda_i \geq \delta_{min}$ for $i = 2, N$. By substituting this fact into (52), we can obtain

$$Q(\grave{A}_i^0)^T + \grave{A}_i^0 Q - k_c\lambda_{p,i}\grave{B}_0\grave{B}_0^T + \frac{1}{\gamma^2}\grave{F}_0\grave{F}_0^T + Q\grave{C}_0^T\grave{C}_0 Q < 0, \text{for } i = 2, N. \tag{53}$$

Since for a connected plant network $\lambda_{p,j} > 0, \forall j = 2, \ldots, N$ and $\lambda_{p,2}$, $\lambda_{p,N}$ are the smallest and largest non zero eigenvalues of $L_p$, then it follows that there exists $\theta \in [0,1]$ such that $\lambda_{p,j} = (1-\theta)\lambda_{p,2} + \theta\lambda_{p,N}$. Thus, it follows from (42) that $\grave{A}_j^0 = (1-\theta)\grave{A}_2^0 + \theta\grave{A}_N^0$. By combining these two facts with (53), we get

$$Q(\grave{A}_j^0)^T + \grave{A}_j^0 Q - k_c\lambda_{p,j}\grave{B}_0\grave{B}_0^T + \frac{1}{\gamma^2}\grave{F}_0\grave{F}_0^T + Q\grave{C}_0^T\grave{C}_0 Q < 0, \forall j = 2,\ldots,N. \quad (54)$$

From (42), we have $\grave{A}_j^0 = \grave{A}_j - k_c\lambda_{p,j}\grave{B}_0\grave{K}_0$. By substituting this into (54), we can obtain

$$Q(\grave{A}_j - k_c\lambda_{p,j}\grave{B}_0\grave{K}_0)^T + (\grave{A}_j - k_c\lambda_{p,j}\grave{B}_0\grave{K}_0)Q - k_c\lambda_{p,j}\grave{B}_0\grave{B}_0^T + \frac{1}{\gamma^2}\grave{F}_0\grave{F}_0^T + Q\grave{C}_0^T\grave{C}_0 Q < 0,$$
$$\forall j = 2,\ldots,N. \quad (55)$$

By substituting (51) into (55), we have

$$Q\grave{A}_j^T + \grave{A}_j Q + \frac{1}{\gamma^2}\grave{F}_0\grave{F}_0^T + Q\grave{C}_0^T\grave{C}_0 Q < 0, \forall j = 2,\ldots,N. \quad (56)$$

Let $P_j = Q^{-1}$ for all $j = 2,3,\cdots,N$, and then we can see that (41) can be equivalently obtained from (56). Based on Lemma 4.1 and using the final decomposed system (44), the closed loop system (29) asymptotically achieves consensus when $w(t) \equiv 0$ and $\|T_{zw}(s)\|_\infty = \max_{j=2,\ldots,N} \left\|\grave{C}_0[sI - (\grave{A}_j^0 + k_c\lambda_{p,j}\grave{B}_0\grave{K}_0)]^{-1}\grave{F}_0\right\|_\infty < \gamma$ when $w(t) \not\equiv 0, \forall j = 2,\ldots,N$. □

Using Theorem 4.2, we can introduce the following steps for solving the $H_\infty$ problem (3.1):

(1) Given the reduced order closed loop system (44), compute the minimum $H_\infty$ attenuation performance $\gamma_{min}$ by solving the following minimization problem:

$$\begin{aligned} &\text{Minimize } \gamma \\ &\text{Subject to LMI (50), with } \gamma > 0,\ Q = Q^T > 0, \text{and } \delta_{min} > 0. \end{aligned} \quad (57)$$

(2) For a given value of the prescribed attenuation performance $\gamma \geq \gamma_{min}$, solve LMIs (50) for a feasible solutions $Q = Q^T > 0$, and $\delta_{min} > 0$. Note that here $\gamma$ is constant and given, thus we solve for a feasible solutions $Q = Q^T > 0$, and $\delta_{min} > 0$ that satisfy (50).
(3) The static gain of the proposed decentralized controller $K$ can be computed as follow: By Theorem 4.2, we have $\grave{K}_0 = -\frac{1}{2}\grave{B}_0^T Q^{-1}$, where $Q$ is computed from the above step, $\grave{B}_0 = \begin{bmatrix} B_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, and $\grave{K}_0 = \begin{bmatrix} K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. Then we can obtain the static controller gain matrix of the proposed decentralized controller (22) $K$ from $\grave{K}_0$.
(4) The scalar parameter $k_c$ of the proposed decentralized controller (22) must be chosen such that $k_c \geq \frac{\delta_{min}}{\lambda_{p,2}}$, where $\delta_{min}$ is computed from step number 2.

## 5 Illustrative Simulations

Given the plant network $G_p$, shown in Fig. Figure 4, assume that the nodes have identical LTI transfer functions ($P_i(s) = P(s), \forall i = 1,\ldots,N = 4$), where

$$P_i(s) = P(s) = \frac{10}{s^2 + 10s + 40}, \forall i = 1,\ldots,N = 4. \quad (58)$$

The dynamics of the nodes can be modeled in matrix form as in (18), where

$$A_p = \begin{bmatrix} -10 & -5 \\ 8 & 0 \end{bmatrix}, B_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_p = \begin{bmatrix} 0 & 1.25 \end{bmatrix}, D_p = 0, \text{ and } F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Assume that the plant network topology has identical LTI edges dynamics ($e_{ij}(s) = \lambda_{ij}(s) = H(s)$, for all $i$, $j$), where

$$H(s) = \frac{s^2 + s + 3}{s^2 + 2s + 1}. \tag{59}$$

The network topology dynamics of the plant network $G_p$ are given by (20), where

$$A_e = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, B_e = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C_e = \begin{bmatrix} -0.5 & 1 \end{bmatrix}, D_e = 1, \text{and } L_p = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

The controlled-output function (disagreement function) $z_i(t)$ is defined for the plant network shown in Fig. Figure 4 as:

$$z_i(t) = \frac{1}{4}\sum_{j=1}^{N}[x_{p,i}(t) - x_{p,j}(t)] = x_{p,i}(t) - \frac{1}{4}\sum_{j=1}^{4} x_{p,j}(t), \forall i = 1, \dots, 4. \tag{60}$$

The $H_\infty$ suboptimal problem (Problem 3.1) can then be solved as follows:

(1) First, we compute the smallest and largest non-zero eigenvalues of $L_p$ of the plant network $G_p$, $\lambda_{p,2} = 2$, $\lambda_{p,N} = 4$ and then we can compute the minimum $H_\infty$ performance $\gamma_{min}$ by solving the minimization problem (57) for the reduced system (44). Note that the input data for the minimization problem (57) are given by:

$$\grave{A}_2^0 = \begin{bmatrix} -10 & -2.5 & -0.5 & 1 \\ 8 & 0 & 0 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \grave{A}_2^0 = \begin{bmatrix} -10 & 0 & -0.5 & 1 \\ 8 & 0 & 0 & 0 \\ 0 & 10 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \grave{B}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \grave{F}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and $\grave{C}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

We use the Yalmip toolbox (Lofberg 2004) with the SeDuMi solver (Sturm 1999) for solving the minimization problem (57). As result we get $\gamma_{min} = 0.125$.

(2) For a given $\gamma \geq \gamma_{min} = 0.125$, we choose $\gamma = 0.5$ and then we solve the LMI (50) for a feasible solutions $Q = diag(Q_1, Q_2) = Q^T > 0$, and $\delta_{min} > 0$. As result we get

$$Q = \begin{bmatrix} 0.9913 & -0.7478 & 0 & 0 \\ -0.7478 & 0.6650 & 0 & 0 \\ 0 & 0 & 5.4563 & -2.7555 \\ 0 & 0 & -2.7555 & 7.0080 \end{bmatrix}, \text{and } \delta_{min} = 37.7805.$$

Note that we do not use $\gamma = \gamma_{min}$ because this will result in too large controller gain.

(3) In this step, we compute the decentralized controller gain $K_0$ as: $\grave{K}_0 = -\frac{1}{2}\grave{B}_0^T Q^{-1}$. Thus,

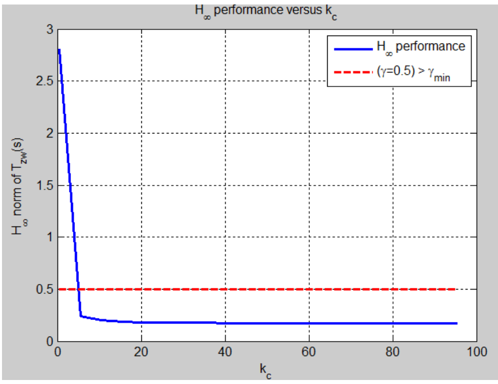


Figure 6: $H_\infty$ performance for different values of $k_c$.

we get

$$\grave{K}_0 = \begin{bmatrix} -3.3254 & -3.7396 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{and then } K = \begin{bmatrix} -3.3254 & -3.7396 \end{bmatrix}.$$

(4) Choose the scalar parameter $k_c$ of the proposed decentralized controller such that $k_c \geq \frac{\delta_{min}}{\lambda_{p,2}}$, which is $k_c \geq \frac{37.7805}{2} = 18.8903$.

This result can be verified by taking the $H_\infty$ norm of the transfer function matrix from $w$ to $z$ (using multi-variable Toolbox with MATLAB, (Boyle et al. 1989)), $T_{zw}(s)$ of the system (44) with the decentralized controller gain $K$ computed above for different values of $k_c$ yields $\left\| \grave{C}_0[sI - (\grave{A}_2^0 + k_c \lambda_{p,2} \grave{B}_0 \grave{K}_0)]^{-1} \grave{F}_0 \right\|_\infty < \gamma$ for different values of $k_c$ as shown in Fig. Figure 6. As we note from the Fig. Figure 6 that when $k_c \geq \frac{\delta_{min}}{\lambda_{p,2}} = 18.8903$ then we guarantee the $H_\infty$ norm of $T_{zw}$ below the prescribed attenuation performance level $\gamma = 0.5$. Note that Fig. 4 in (Li et al. 2011) represents the $H_\infty$ performance region (see Definition 1 in (Li et al. 2011)) for the static case. Similarly, the $H_\infty$ performance region in this case can be computed from Fig. Figure 6 as the region $[18.8903, \infty)$. In addition, in this region we guarantee the $H_\infty$ norm of $T_{zw}$ below the prescribed value $\gamma = 0.5$.

Note that we can also design the controller parameters ($K$, and $k_c$) that are associated with the minimum $H_\infty$ performance $\gamma_{min}$ by solving the minimization problem (57) for the reduced system (29). Thus we get

$$Q = 10^4 \times \begin{bmatrix} 0.0472 & -0.0008 & 0 & 0 \\ -0.0008 & 0.00001 & 0 & 0 \\ 0 & 0 & 1.9318 & -1.0899 \\ 0 & 0 & -1.0899 & 2.1487 \end{bmatrix},$$

$$\grave{K}_0 = -\frac{1}{2}\grave{B}_0^T Q^{-1} = \begin{bmatrix} -0.0304 & -1.7076 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{and then } K = [\,-0.0304, \ -1.7076].$$

Finally, choose the scalar parameter $k_c$ of the proposed decentralized controller such that $k_c \geq \frac{\delta_{min}}{\lambda_{p,2}} = \frac{3.077\times 10^5}{2} = 1.5385 \times 10^5$ to guarantee the $H_\infty$ norm of $T_{zw}$ below the minimum value $\gamma_{min} = 0.125$. Using the lower bound of the scalar parameter $k_c$, we can rewrite the decentralized

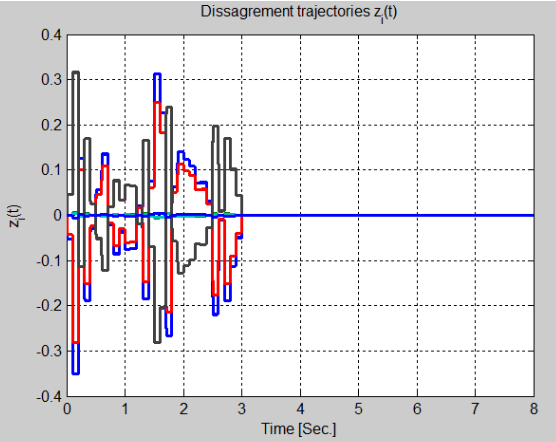


Figure 7: Disagreement trajectories $z_i(t)$ of the plant nodes.

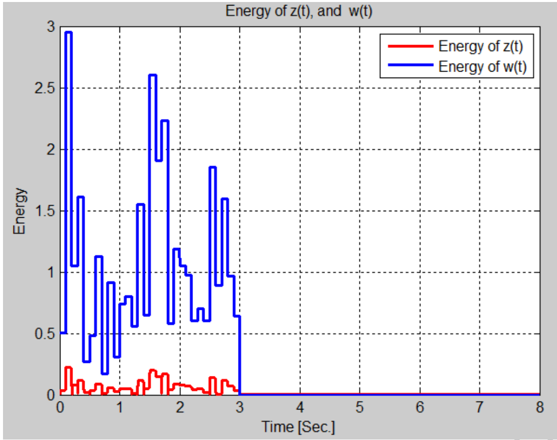


Figure 8: Energy of the disagreement trajectory $z(t)$ and disturbance trajectory $w(t)$.

controller (21) as:

$$u^{in}(t) = (\frac{\delta_{min}}{\lambda_{p,2}})(L_p \otimes K)x_p(t), \tag{61}$$

Using the decentralized controller (61), we can guarantee that the $H_\infty$ norm of $T_{zw}$ equal to the minimum value $\gamma_{min} = 0.125$. This also can be verified by subsisting the new controller formula (61) into the original closed loop system (29), which gives

$$\begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_e(t) \end{bmatrix} = \begin{bmatrix} I_N \otimes A_p + L_p \otimes B_p D_e C_p + \frac{\delta_{min}}{\lambda_{p,2}}(L_p \otimes B_p K) & I_N \otimes B_p C_e \\ L_p \otimes B_e C_p & I_N \otimes A_e \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_e(t) \end{bmatrix} + \begin{bmatrix} I_N \otimes F \\ \mathbf{0} \end{bmatrix} w(t)$$
$$z(t) = \begin{bmatrix} L_g \otimes I_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_e(t) \end{bmatrix}, \tag{62}$$

where $L_g = \frac{1}{N} L_p = \frac{1}{4} L_p$.

Taking the $H_\infty$ norm of the transfer function matrix from $w$ to $z$ (using the multi-variable Toolbox with MATLAB, (Boyle et al. 1989)), $T_{zw}(s)$ of the system (62) with the decentralized controller gain $K = [\,-0.0304, \ -1.7076]$, $\lambda_{p,2} = 2$ and $\delta_{min} = 1.5385 \times 10^5$. As a result we get the $H_\infty$ norm of $T_{zw}$, $\|T_{zw}(s)\|_\infty = 0.125$, which is the same as $\gamma_{min} = 0.125$ computed above using the reduced, decomposed system, which validates and demonstrates the correctness of the decomposition procedure and the algorithm developed in this chapter for solving the $H_\infty$ suboptimal control problem.

To see the result in simulation we apply a band-limited white noise $w(t)$ to disturb the system (62) during the first three seconds with the parameters computed above. The simulation result of the disagreement $z_i(t)$ and the energy (root mean square) of the disagreement trajectory $z(t)$ and disturbance trajectory $w(t)$ are shown in Fig. Figure 7 and Fig. Figure 8, restrictively.

Note that we used a root mean square RMS block with a Matlab Simulink to measure the energy of the disagreement trajectory $z(t)$ and disturbance trajectory $w(t)$ as shown in Fig. Figure 8. We can note from these figures that the consensus is asymptotically achieved when $w(t) \equiv 0$ (after three seconds) and $\|T_{zw}(s)\|_\infty$ is minimized and then the energy is also minimized. Thus we verify that the $H_\infty$ norm of the closed-loop transfer function matrix is an upper bound on the amplification of the energy in the disturbance vector $w$ to the disagreement vector $z$.

## 6 Conclusions

This paper considered the problem of minimizing deviation from consensus in the presence of disturbances. A decentralized controller was proposed, then model, orthogonal and diagonal transformations were applied to our model in order to reduce the overall system into $N-1$ independent subsystems, and finally conditions were derived in terms of LMIs for the network to achieve consensus in the absence of disturbances and attenuation of disturbances to a prescribed $H_\infty$ performance level. An algorithm was provided for solving the $H_\infty$ suboptimal control problem for a dynamic network with identical LTI nodes and edges. An example was given to illustrate and verify these results.

As future work, the controller design problem addressed herein assumes dynamic networks with identical dynamics for nodes and edges. It would be interesting to address the $H_\infty$ suboptimal control problem of dynamic networks with non-identical dynamics for nodes and edges. This more general problem cannot be decomposed into lower-order systems, which presents a complex challenge. An additional research direction would be an extension to the $H_2$ suboptimal control problem.

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